SOME NOTES ON INTEGRAL EQUIVALENCE OF COMBINATORIAL MATRICES

BY
W. D. WALLIS

ABSTRACT

Recent work on integral equivalence of Hadamard matrices and block designs is generalized in two directions. We first determine the two greatest invariants under integral equivalence of the incidence matrix of a symmetric balanced incomplete block design. This enables us to write down all the invariants in the case where $k - \lambda$ is square-free. Some other results on the sequence of invariants are presented. Secondly we consider the existence of inequivalent Hadamard matrices under integral equivalence. We show that if there is a skew-Hadamard matrix of order $8m$ then there are two inequivalent Hadamard matrices of order $16m$, that and there are precisely eleven inequivalent Hadamard matrices of order 32.

1. We assume the standard definitions of an Hadamard matrix as a square $(1, -1)$ matrix whose rows (and consequently whose columns) are mutually orthogonal, and of a skew-Hadamard matrix as an Hadamard matrix of the form $I + S$, where S is skew. An Hadamard matrix necessarily has order 1, 2, or a multiple of 4.

Two integral matrices A and B are (integrally) equivalent if there exist integer matrices P and Q of determinants ± 1 which satisfy

$$A = PBQ.$$

If A has rank r then A determines a unique set of positive integers $a_1, a_2, \cdots, a_r$, the (integral equivalence) invariants of A, such that a_i divides a_{i+1} for $i = 1, 2, \cdots, r-1$, and such that A is equivalent to the canonical diagonal matrix (Smith normal form)

$$D = \operatorname{diag}(a_1, a_2, \cdots, a_r, 0, \cdots 0);$$

Received (in two parts) on May 10, 1971 and June 8, 1971. Received in revised form September 9, 1971

two matrices are equivalent if and only if they have the same set of invariants. (These results can be found in various standard texts.)

In particular, if A is an Hadamard matrix of order n, the invariant sequence has the property

$$a_i a_{n-i+1} = n;$$

$a_1 = 1$, and the next t invariants equal 2 where

$$t \geqq [\log_2 (n-1)] + 1$$

(see [5, 6, 7]). From this we can find an upper bound on the number of inequivalent Hadamard matrices of a given order.

In this paper we extend the known results in two directions. In Section 2 we consider the equivalence invariants of the incidence matrix of a symmetric balanced incomplete block design, as suggested by Newman [5], and results like those above are obtained. In the rest of the paper we discuss the number of inequivalent Hadamard matrices: we prove in Section 3 that there are at least two inequivalent Hadamard matrices of order $16m$ when there is a skew-Hadamard matrix of order $8m$, and in Section 4 that there are precisely 11 Hadamard matrices of order 32. (The latter result was conjectured in [7].) We use the fact, reported in [1], that if A and B are Hadamard matrices of order n then

$$\begin{pmatrix} A & B \\ -A & B \end{pmatrix}$$

is an Hadamard matrix of order $2n$.

The most natural equivalence relation to use in discussing Hadamard matrices is Hadamard equivalence, under which two matrices are considered equivalent if one can be obtained from the other by a sequence of row interchanges, column interchanges, row negations and column negations. Inequivalent matrices are Hadamard-inequivalent, but the converse does not hold (see [7]); therefore our results give *lower* bounds on the number of Hadamard-inequivalent matrices.

Unless otherwise specified, we follow the notation of [7].

2. Define A to be the incidence matrix of a symmetric balanced incomplete block design with parameters (v, k, λ), where $k > \lambda$, and write $a_1, a_2, \cdots, a_v$ for the integral equivalence invariants of A and D for the canonical matrix of A. We write $n = k - \lambda$.

We prove the following propositions:

THEOREM 1. *The first* $[\log_2 v] + 1$ *invariants of* A *equal* 1 *and the last two are* n *and* $nk(k,\lambda)^{-1}$. *If* n *is square-free the invariants are**

1	$(v-1)/2$ *times*
(k,λ)	*once*
n	$(v-3)/2$ *times*
$nk(k,\lambda)^{-1}$	*once.*

THEOREM 2. *If the* s *consecutive invariants* $(s \geqq 3)$ *starting from* a_i *are equal, then the* $s-2$ *consecutive invariants ending at* a_{v+i} *are each equal to* na_i^{-1}.

We need the following lemma.

LEMMA 1. *Suppose* L *is a* $v \times v$ *matrix over a Euclidean domain* E *and* y *is a member of* E. *Write* $l_1, l_2, \cdots l_v$ *for the invariants of* L *and* $m_1, m_2, \cdots, m_v$ *for the invariants of* $M = L + yJ$. Then

$$l_{i-1} \mid m_i \qquad i = 2, 3, \cdots, v;$$
$$m_{i-1} \mid l_i \qquad i = 2, 3, \cdots, v.$$

PROOF. The typical $i \times i$ submatrix of M has the form $X + yJ$, where X is an $i \times i$ submatrix of L. Write X_j for X with the jth column replaced by $y\varepsilon_i$. Then

$$|X + yJ| = |X| + \sum_{j=1}^{i} |X_j| + T \tag{1}$$

where T is a sum of determinants with two columns $y\varepsilon_i$, and so equals zero. Expanding $|X_j|$ by its jth column we obtain y times a sum of determinants of $(i-1) \times (i-1)$ submatrices of L. So l_{i-1} divides each term on the right-hand side of (1). Therefore

$$l_{i-1} \mid m_i \qquad i = 2, 3, \cdots, v. \tag{2}$$

Similarly, since $L = M + (-y)J$,

$$m_{i-1} \mid l_i. \tag{3}$$

The adjoint of A is

$$n^{(v-3)/2}(kA^T - \lambda J)$$

[3, p. 25], so $B = kA^T - \lambda J$ satisfies

* This canonical form was found in [3,5] for the case $(k,\lambda) = 1$.

$$AB = knI.$$

If P and Q are unimodular matrices such that $PAQ = D$, then

$$Q^{-1}BP^{-1} = knD^{-1}$$

which is necessarily the canonical matrix of B with its entries in reverse order. (It is diagonal, and each diagonal entry divides each earlier one.) So the invariants $b_1, b_2, \cdots, b_v$ of B satisfy

(4) $$b_i a_{v-i+1} = kn.$$

The non-zero entries in B equal n or λ, so b_1 is the greatest common divisor $(n,\lambda) = (k,\lambda)$, and

(5) $$a_v = nk(k,\lambda)^{-1}$$

The invariants of kA^T are $ka_1, ka_2, \cdots, ka_v$, so from (1) and (2)

(6) $$ka_{i-1} \mid b_i \qquad 2 \leqq i \leqq v,$$

(7) $$b_{i-1} \mid ka_i \qquad 2 \leqq i \leqq v.$$

Suppose $a_i = a_{i+1} = \cdots = a_{i+s-1}$, where $s \geqq 3$. Then $b_{v-i+1} = b_{v-i} = \cdots = b_{v-i-s+2} = nka_i^{-1}$, by (4). From (6) and (7)

$$ka_{v-i} \mid nka_i^{-1},$$

$$nka_i^{-1} \mid ka_{v-i-s+3},$$

and since

$$a_{v-i-s+3} \mid a_{v-i}$$

equality must hold throughout, that is

$$a_{v-i-s+3} = a_{v-i-s+4} = \cdots = a_{v-i} = na^{-1},$$

which proves Theorem 2.

We know [7, Theorem 3] that A has at least $[\log_2 v] + 1$ invariants equal to 1. Suppose $v \geqq 4$. Then $a_1 = a_2 = a_3 = 1$, so $a_{v-1} = n$. In the (all trivial) cases where $v < 4$ this is also true. So we have the first part of Theorem 1; if n is square-free the rest of Theorem 1 is easy to prove.

It is worth observing that the conditions derived here are certainly not sufficient. For example, the Theorems would allow four possible invariant sequences for a $(16,6,2)$ design: the first a invariants equal 1, $16 - 2a$ equal 2, $a - 1$ equal 4 and one equals 12, where $a = 5,6,7$ and 8. The possible $(16,6,2)$ designs are exhibited in [2], and on testing we find that only the cases $a = 6$ and $a = 7$ occur.

3. LEMMA 2. *If there is a skew-Hadamard matrix of order n then there is a skew-Hadamard matrix of order 2n with invariants*

1	*once*
2	$n-1$ *times*
n	$n-1$ *times*
$2n$	*once.*

PROOF. The Theorem is easily proven when $n = 1$ or 2, so put $n = 4m$. Suppose A is a skew-Hadamard matrix with canonical diagonal matrix D; suppose P and Q are unimodular integral matrices such that

$$D = PAQ.$$

Then

$$Q^{-1}A^TP^{-1} = nD^{-1},$$

and nD^{-1} is the matrix D with the order of its entries reversed [5]. For convenience write

$$D = (1) \oplus 2C \oplus (4m);$$

C is a diagonal integral matrix of order $n-2$.

Consider the matrix

$$K = \begin{pmatrix} A & A \\ -A^T & A^T \end{pmatrix}$$

which is skew-Hadamard of order $2n$. K is equivalent to

$$\begin{pmatrix} P & 0 \\ Q^{-1} & -Q^{-1} \end{pmatrix} \begin{pmatrix} A & A \\ -A^T & A^T \end{pmatrix} \begin{pmatrix} Q & P^{-1} \\ 0 & P^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} PAQ & 0 \\ Q^{-1}(A+A^T)Q & 2Q^{-1}A^TP^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} D & 0 \\ 2I & 2nD^{-1} \end{pmatrix}$$

using the fact that $A + A^T = 2I$. This last matrix is

$$
\left(\begin{array}{c|c|c|c|c|c}
1 & & & & & \\ \hline
 & 2C & & & & \\ \hline
 & & n & & & \\ \hline
2 & & & 2n & & \\ \hline
 & I & & & nC^{-1} & \\ \hline
 & & 2 & & & 2
\end{array}\right)
$$

Subtract twice row 1 from row $n+1$; subtract column $2n$ from column n; then we can reorder the columns and rows to obtain

$$
\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & n & \\ & & & 2n \end{bmatrix} \oplus \left(\begin{array}{c|c} 2C & 0 \\ \hline 2I & nC^{-1} \end{array}\right).
$$

Every entry of C divides $\frac{1}{2}n$, so $\frac{1}{2}nC^{-1}$ is integral. So the second direct summand is integrally equivalent to

$$
\begin{pmatrix} -I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} 2C & 0 \\ 2I & nC^{-1} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}nC^{-1} & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} nI & 0 \\ 0 & 2I \end{pmatrix}
$$

and the invariants of K are as required.

LEMMA 3. *If there is an Hadamard matrix of order $n = 8m$, then there is an Hadamard matrix of order $2n$ with at least $12m - 1$ invariants divisible by 4.*

PROOF. If A is Hadamard of order n and has canonical diagonal matrix D, then

$$
H = \begin{pmatrix} A & A \\ -A & A \end{pmatrix}
$$

is Hadamard of order $2n$ and is equivalent to the diagonal matrix

$$
D \oplus 2D.
$$

Since $n = 8m$, the last $4m$ invariants of A are divisible by 4. Every entry of $2D$ except the first is divisible by 4. So $D \oplus 2D$ has at least $12m - 1$ entries divisible by 4. Even if $D \oplus 2D$ is not in canonical form, it is easy to deduce that $D \oplus 2D$ (and consequently H) has at least $12m - 1$ invariants divisible by 4.

THEOREM 3. *If there is a skew-Hadamard matrix of order $8m$ then there exists a pair of inequivalent Hadamard matrices of order $16m$.*

PROOF. Lemmas 2 and 3.

4. The restrictions in Section 1 imply that the invariants of an Hadamard matrix of order 32 are

1	*once*
2	*a times*
4	$(15 - a)$ *times*
8	$(15 - a)$ *times*
16	*a times*
32	*once*

and that $5 \leqq a \leqq 15$. We shall say a matrix with the invariants shown is "of class a"; we shall now construct Hadamard matrices of order 32 in all eleven possible classes, thus proving

THEOREM 4. *There are precisely eleven inequivalent Hadamard matrices of order* 32.

In Section 4 of [7] we calculated the invariants of some Hadamard matrices of order a power of 2 by generating functions. In particular, if A is Hadamard of order 16 with ω invariants equal to 2 and if H_2 is Hadamard of order 2, the direct product

$$H_2 \times A = \begin{pmatrix} A & A \\ -A & A \end{pmatrix}$$

has exactly $\omega + 1$ invariants equal to 2. So the existence of 16×16 Hadamard matrices with 4, 5, 6 and 7 invariants equal to 2 (exhibited in [7]) implies the existence of 32×32 Hadamard matrices of classes 5, 6, 7 and 8. There is a skew-Hadamard matrix of order 16, so by Lemma 2 class 15 exists.

An Hadamard matrix of order 16 can be constructed from a symmetric balanced incomplete block design with parameters (15, 7, 3): first construct a matrix with (i, j) entries 1 if treatment j belongs to block i and -1 elsewhere; then add on a first row and column with every entry 1. The (15, 7, 3)-designs have been found by Nandi [4], and are also listed in [1]. Write A for the 16×16 Hadamard matrix constructed from Nandi's design $(a_2 a_2')$, and B for the matrix constructed from Nandi's $(a_1 a_1')_1$ after applying a permutation π to the blocks. Then consider the matrix

$$H = \begin{pmatrix} A & B \\ -A & B \end{pmatrix}.$$

It is found that

when $\pi = (1)$	H *is of class* 9,
when $\pi = (1,3)$	H *is of class* 10,
when $\pi = (1,3,4)$	H *is of class* 11,
when $\pi = (2,7,12,13)$	H *is of class* 12,
when $\pi = (3,4,5,6,7)$	H *is of class* 13,
when $\pi = (2,3,4,5,6,7)$	H *is of class* 14.

Therefore, examples of all classes can be found. (These results were found in a computer test of various 32×32 Hadamard matrices).

It should be observed that these results could be used to strengthen the lower bounds found in [7] on the number of inequivalent Hadamard matrices of order a power of 2.

References

1. Vasanti N. Bhat and S. S. Shrikhande, *Non-isomorphic solutions of some balanced incomplete block designs*, I., J. Combinatorial Theory, **9** (1970), 174–191.

2. Q. M. Husain, *On the totality of the solutions for the symmetrical balanced incomplete block designs:* $\lambda = 2$, $k = 5$ *or* 6, Sankhyâ 7 (1945), 204–208.

3. M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, 1964.

4. H. K. Nandi, *A further note on non-isomorphic solutions of incomplete block designs*, Sankhyâ, 7 (1945–46), 313–316.

5. M. Newman, *Invariant factors of combinatorial matrices*, Israel J. Math. **10** (1971), 126–130.

6. W. D. Wallis and Jennifer Wallis, *Equivalence of Hadamard matrices*, Israel J. Math. **7** (1969), 122–128.

7. W. D. Wallis, *Integral equivalence of Hadamard matrices*, Israel J. Math. **10** (1971), 349–358.

Faculty of Mathematics
University of Newcastle
New South Wales, Australia, 2308